

ABSTRACT

on the dissertation of **Bekenayeva Kymbat Slamovna**
submitted for the degree of Doctor of Philosophy (PhD)
in the educational program 8D05401 – Mathematics

Research title: Solvability of initial boundary value problems for a pseudo-parabolic equation of fractional order.

The purpose of the research: The study of the issues of unambiguous solvability of initial boundary value problems for a quasi-linear pseudo-parabolic fractional equation with linear and nonlinear boundary conditions. Establishing a theorem on the existence and uniqueness of a weak generalized solution of problems, proving the blowing-up of the solution to the problem for a quasi-linear pseudo-parabolic fractional equation with a linear boundary condition, and studying the asymptotic behavior of the solution in time.

Research objectives.

For a problem with a linear boundary condition for a pseudo-parabolic equation with a fractional Caputo derivative:

– formulation and investigation of the problem solvability for various cases of the degree of positive constants p and q :

case I, when $0 < q \leq 1, 2 < p < \frac{2N}{N-2}$ for all $N \geq 3$;

case II, when $1 \leq q \leq 1 + \frac{2}{N}, 2 < p < \frac{2N}{N-2}$ for all $N \geq 3$;

case III, when $0 < q \leq 1, 1 < p \leq 2$ for all $N \geq 3$;

case IV, when $1 \leq q \leq 1 + \frac{2}{N}, 1 < p \leq 2$ for all $N \geq 3$;

– proof of the existence of a weak generalized solution to the problem for cases I - IV;

– proof of the uniqueness of the weak solution to the problem under consideration for the case II;

– blowing-up of the solution to the problem in finite time;

– proof of the solvability to the problem for any finite time t for $1 < p \leq 2$;

– study of the asymptotic behavior of a solution in time $t \rightarrow \infty$.

For a problem with a nonlinear boundary condition for a pseudo-parabolic equation with a fractional Caputo derivative:

– formulation and investigation of the unambiguous solvability of the boundary value problem in the presence of nonlinearity both in the equation itself and in the boundary condition;

– proof of local solvability of the problem for the case
 $2 < p < \frac{2n}{n-2}, 2 < \sigma < \frac{2(n-1)}{n-2}, n \geq 3$;

– proof of the uniqueness of the local solution of the problem for
 $2 < p < \frac{2n}{n-2}, n \geq 3$;

– proof of the uniqueness of the global solution of the problem for $1 < p \leq 2$, and additionally, a theorem on the existence of a global solution to the problem for $1 < \sigma < \frac{2(n-1)}{n-2}, n \geq 3$;

– formulation and study the solvability of the initial boundary value problem for a linear pseudo-parabolic loaded fractional order equation with respect to a spatial variable.

Methods of investigation. In the process of achieving the results of the study, the Galerkin approximation method, the method of a priori estimates, the theory of Sobolev spaces, methods of integral and differential calculus, the method of functional analysis, the method of compactness, the method of monotonicity, the method of inequalities for derivatives of fractional order, analytical and functional methods for fractional calculus were applied.

The main provisions submitted for defense (proven scientific hypotheses and also conclusions that are new knowledge). According to the results of the study, the following provisions are submitted for thesis presentation:

1) Establishment and proof of a theorem on the existence and uniqueness of a weak generalized solution to a problem with a linear boundary condition for a quasi-linear pseudo-parabolic equation with a fractional derivative of a Caputo.

2) Proof of the blowing-up the solution to the problem in finite time.

3) The study of the asymptotic behavior of the solution in time.

4) Formulation and proof of the theorem on the existence and uniqueness of a weak solution to a problem with a nonlinear boundary condition for a pseudoparabolic equation with Caputo fractional derivative.

5) Proof of the solvability of the initial boundary value problem for a loaded pseudo-parabolic equation with a fractional Caputo derivative.

The main results of the study.

1. The solvability of an initial boundary value problem with a nonlinear boundary condition for a pseudo-parabolic equation with a fractional Caputo derivative is considered. The existence of a weak solution is investigated using Galerkin approximations and a priori estimates, the uniqueness of the weak solution of the problem is proved by applying the Sobolev embedding theorem, Rellich-Kondrashov theorem and the Gronwall-Bellman lemma. The analysis is also carried out and the blowing-up the solution to the problem in finite time is proved, that the solution itself tends to infinity for $t \rightarrow T^*$ in a certain set Ω of x values. The global solvability of the initial boundary value problem and the uniqueness of the weak generalized solution are studied.

2. The solvability of initial boundary value problems for a quasi-linear pseudo-parabolic equation of fractional order with a sufficiently smooth boundary is investigated. The difference between the studied problems lies in the fact that the boundary conditions are set in the form of a nonlinear boundary condition with a fractional differentiation operator. The main result is the establishment of local or global solvability of the problems, depending on the parameters of the equation. Using the Galerkin method, the existence of a weak solution of a quasi-linear pseudo-parabolic equation in a bounded domain is proved. Using Sobolev embedding theorems, a priori estimates of the solution are obtained. A priori estimates and the Rellich-Kondrashov theorem are used to prove

the existence of the desired solutions to the boundary value problems under consideration. The uniqueness of weak generalized solutions to initial boundary value problems is proved on the basis of the obtained a priori estimates and the application of the generalized Gronwall Lemma.

3. An initial boundary value problem for a loaded pseudo-parabolic equation with a fractional Caputo derivative is considered. The theorem of the existence of the solution to the problem is formulated and proved.

Novelty and importance of the results obtained:

The *substantiation of the first results novelty* of scientific research consists in the fact that in this paper the issues of unambiguous solvability of initial boundary value problems for a quasi-linear pseudo-parabolic fractional equation with linear and nonlinear boundary conditions with fractional Caputo differentiation operators are studied and proved: the theorem of the existence of a weak solution of the problem for a quasi-linear pseudo-parabolic equation with a linear boundary condition for cases I-IV, the uniqueness theorem of the weak solution of the problem for a pseudoparabolic equation with a linear condition for case II, destruction of the solution in finite time; the asymptotic behavior of the solution in time is investigated.

The *substantiation of the second results novelty* of the dissertation work is the formulation and study of the issues of unambiguous solvability of boundary value problems, the difference of which lies in the fact that fractional differentiation operators participate both in the equation itself and in the boundary condition in the form of a nonlinear condition; for the case $2 < p < \frac{2n}{n-2}$, $2 < \sigma < \frac{2(n-1)}{n-2}$, $n \geq 3$, the theorem on the local solvability of the problem is proved; the uniqueness theorem of the local solution to the problem is proved, if the following conditions hold $2 < p < \frac{2n}{n-2}$, $n \geq 3$; the theorem on the uniqueness of the global solution to the problem is proved under the condition $1 < p \leq 2$, and additionally, the theorem on the existence of a global solution to the problem when the conditions are met $1 < \sigma < \frac{2(n-1)}{n-2}$, $n \geq 3$.

The *substantiation of the third results novelty* is formulation and proof the theorem of the existence of the solution to the initial boundary value problem for a loaded pseudo-parabolic equation with a fractional Caputo derivative. The problem for a loaded equation with respect to a spatial variable for a linear pseudo-parabolic equation with initial and second boundary conditions is reduced to a loaded equation with a non-local boundary condition. The solvability of the problem is proved using the continuation method on a parameter.

The main provisions were obtained by detailed proofs given in the research paper based on the results previously obtained by well-known scientists, and the reliability and novelty of the results of the work can be confirmed by their publication in rating peer-reviewed publications included in the international scient metric databases Web of Science Core Collection and Scopus.

Compliance with the directions of science development or state programs:

The State Program for the Development of Education and Science of the Republic of Kazakhstan for 2020 - 2025 (No. 988 dated December 27, 2019); Concepts for the development of Science of the Republic of Kazakhstan for 2022 - 2026 (Resolution of the Government of the Republic of Kazakhstan No. 336 dated May 25, 2022).

The contribution of the doctoral student to the preparation of each publication (the contribution of the author of the dissertation is shown as a percentage of the total volume of the publication):

4 scientific papers have been published on the topic of the dissertation, 2 of them in foreign publications included in the Scopus and Web of Science databases:

1. Solvability Issues of a Pseudo-Parabolic Fractional Order Equation with a Nonlinear Boundary Condition // Fractal and Fractional – 2021. Vol 5, No. 4, 134; (Co-authors: Aitzhanov, S.E., Berdyshev, A.S., 60%) (IF – 3.167; WoS – Q1, Scopus – 87). doi: <https://doi.org/10.3390/fractalfract5040134>. In the paper, PhD candidate proved the existence and uniqueness theorems of a local generalized solution for a quasi-linear pseudo-parabolic fractional order equation with a nonlinear condition;

2. Solvability of pseudoparabolic equation with Caputo fractional derivative // Chaos, Solitons and Fractals – 2022, Vol 160, 112193; (Co-authors: Aitzhanov, S.E., Kuserbayeva U.R., 85%) (IF – 9.92; WoS – Q1, Scopus – 99). <https://doi.org/10.1016/j.chaos.2022.112193>. In the article, PhD candidate has proved theorems on the existence and uniqueness of a weak generalized solution of a problem with a linear boundary condition for a pseudo-parabolic equation with a fractional Caputo derivative, and also proved the blowing-up of the solution to the problem in finite time;

3. Boundary value problems for pseudo-parabolic equation with fractional order derivatives, Abstracts of the Uzbekistan-Malaysia conference «Computational models and technologies», September 16-17th, 2022, P.142-143. (Co-authors: Aitzhanov S.E., Berdyshev A.S., 90%);

4. Solvability of the Sobolev equation with a fractional Caputo derivative, Abstracts of the International Scientific Conference named «Non-classical equations of mathematical physics», Tashkent, October 6-8, 2022, C.78-79. (Co-author: Baishemirov Zh.D., 90%) (in Russian).